Analysis UROP Project Summary
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Contents

1 The Cauchy-Schwarz Inequality

This document was made to summarize all the key parts of this UROP project. This includes theorems and exercises from the book *A View from the Top* by Alex Iosevich, and general notes and pending questions from the UROP. For the first five weeks of this UROP, from June 03-July 07, Yuqiu and I read through Chapters 1-8 of Iosevich's text, and from then on we started considering questions brought up Section 2. The first place we can and will start off in these notes however is with the CS inequality (as the title of this Section implies).

1.1 The Inequality Itself

We will prove this inequality in a few ways, both of which are outlined in Iosevich's text, either directly or in an exercise.

Let $a, b \in \mathbb{R}$. Then,

$$\begin{aligned} (a-b)^2 &\geq 0 \\ a^2 - 2ab + b^2 &\geq 0 \\ \implies ab &\leq \frac{a^2+b^2}{2}. \end{aligned}$$

Hence, consider the finite sums

$$A_N = \sum_{k=1}^{N} a_k \quad \text{and} \quad B_N = \sum_{k=1}^{N} b_k$$

where $a_i, b_i \in \mathbb{R}$ for all $1 \leq i \leq N$. To simplify terms long term, let

$$X_N = \left(\sum_{k=1}^{N} a_k^2 \right)^{\frac{1}{2}} \quad \text{and} \quad Y_N = \left(\sum_{k=1}^{N} b_k^2 \right)^{\frac{1}{2}}.$$

Note that X_N and Y_N are just constants! Then, we get the following:

$$\sum_{k=1}^{N} a_k b_k = X_N Y_N \sum_{k=1}^{N} \frac{a_n}{X_N} \cdot \frac{b_k}{Y_N}.$$

Using the fact that $ab \leq \frac{a^2+b^2}{2}$ for all $a, b \in \mathbb{R}$,

$$\begin{aligned} \#S_N = \sum_{k=1}^{N} a_k b_k &\leq X_N Y_N \cdot \left(\sum_{k=1}^{N} \frac{1}{2} \cdot \left(\frac{a_k}{X_N} \right)^2 + \frac{1}{2} \cdot \left(\frac{b_k}{Y_N} \right)^2 \right) \\ &= \frac{X_N Y_N}{2X_N^2} \left(\sum_{k=1}^{N} a_k^2 \right) + \frac{X_N Y_N}{2Y_N^2} \left(\sum_{k=1}^{N} b_k^2 \right) \\ &= \frac{X_N Y_N}{2X_N^2} \left(X_N^2 \right) + \frac{X_N Y_N}{2Y_N^2} \left(Y_N^2 \right) \\ &= X_N Y_N \\ &= \left(\sum_{k=1}^{N} a_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{N} b_k^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Theorem 1 (The Cauchy-Schwarz Inequality)
Therefore, we have

$$\sum_{k=1}^{N} a_k b_k \leq \left(\sum_{k=1}^{N} a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{N} b_k^2\right)^{\frac{1}{2}}. \tag{1}$$

For another way to prove this inequality, consider the standard $\mathbb{R}^n$ Hermitian inner product $\langle a - tb, a - tb\rangle$ for $t \in [0, 1]$. We will minimize this inner product using calculus:

$$\begin{aligned}\langle a - tb, a - tb\rangle &= (a_1 - tb_1)^2 + (a_2 - tb_2)^2 + \cdots + (a_N - tb_N)^2 \\ &= \sum_{k=1}^{N} a_k^2 - 2t\sum_{k=1}^{N} a_k b_k + t^2 \sum_{k=1}^{M} b_k^2 \\ &= ||a||^2 + t^2||b||^2 - 2t\langle a, b\rangle.\end{aligned}$$

It is evident that the critical point of this equation is located at $t = \frac{\langle a,b\rangle}{||b||^2}$. It is furthermore clear that this is where the minimum of the equation is. Therefore, we get the following minimum value:

$$\left\langle a - \frac{\langle a, b\rangle}{||b||^2} b, a - \frac{\langle a, b\rangle}{||b||^2} b \right\rangle = ||a||^2 - \frac{\langle a, b\rangle^2}{||b||^2}.$$

Multiplying by $||b||^2$ on both sides, we can conclude

$$\begin{aligned} 0 &\leq \big|\big| ||b||^2 a - \langle a, b\rangle b \big|\big|^2 \\ &= \big\langle ||b||^2 a - \langle a, b\rangle b, ||b||^2 a - \langle a, b\rangle b \big\rangle \\ &= ||a||^2 ||b||^2 - \langle a, b\rangle^2 \\ \implies \langle a, b\rangle &\leq ||a|| \cdot ||b||. \end{aligned}$$

□

In Chapter 2, Iosevich begins to outline more of what we will be utilizing for this project: projections in $\mathbb{R}^2$ and $\mathbb{R}^3$ (which could extend over to $\mathbb{R}^d$). He does so to begin using the CS inequality in a cool way. For both $\mathbb{R}^2$ and $\mathbb{R}^3$, I will write out the proof that Iosevich uses for the discrete case, and then prove the same inequalities for the "continuous case".

1.2 2D Projections

Let S_N be a set of N points in $\mathbb{R}^2$. Furthermore, let $\pi_1(x_1, x_2) = x_2$ and $\pi_2(x_1, x_2) = x_1$ where $(x_1, x_2) \in \mathbb{R}^2$. Then,

$$\begin{aligned}\sum_{x_1,x_2} \chi_{S_N}(x_1, x_2) &\leq \sum_{x_1,x_2} \chi_{\pi_1(S_N)}(x_2) \cdot \chi_{\pi_2(S_N)}(x_1) \\ &= \sum_{x_1} \chi_{\pi_2(S_N)}(x_1) \cdot \sum_{x_2} \chi_{\pi_1(S_N)}(x_2) \\ &= \#\pi_1(S_N) \cdot \#\pi_2(S_N).\end{aligned}$$

From here you can state that $N^{\frac{1}{2}} \leq \max_{i=1,2} \#\pi_i(S_N)$, but this isn't as important for this project.

Now, instead of considering N points in $\mathbb{R}^2$, lets try to make this more general. Let $\Omega \subset \mathbb{R}^2$, can we say something similar for this case? Well, the answer ends up being yes, using similar logic:

$$\begin{aligned}
\operatorname{area}(\Omega) &= \int_{\mathbb{R}^2} \chi_\Omega(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&\le \int_{\mathbb{R}^2} \chi_{\pi_1(\Omega)}(x_2) \cdot \chi_{\pi_2(\Omega)}(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \int_{\mathbb{R}} \chi_{\pi_2(\Omega)}(x_1)\,\mathrm{d}x_1 \cdot \int_{\mathbb{R}} \chi_{\pi_1(\Omega)}(x_2)\,\mathrm{d}x_2 \\
&= \mathrm{m}(\pi_1(\Omega)) \cdot \mathrm{m}(\pi_2(\Omega))
\end{aligned}$$

where m is the measure (i.e. the length of the subset of $\mathbb{R}$). We get a very useful equation for this project: for all $\Omega \subset \mathbb{R}^2$,

$$\operatorname{area}(\Omega) \le \mathrm{m}(\pi_1(\Omega)) \cdot \mathrm{m}(\pi_2(\Omega)). \tag{2}$$

We can prove this one more way: geometrically. Let $X = \pi_2(\Omega)$ and $Y = \pi_1(\Omega)$ and consider

$$X \times Y := \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

Then, it is clear that $\Omega \subset X \times Y$, and then $\operatorname{area}(\Omega) \le \operatorname{area}(X \times Y)$. Hence,

$$\operatorname{area}(\Omega) \le \operatorname{area}(X \times Y) = \mathrm{m}(X) \cdot \mathrm{m}(Y) = \mathrm{m}(\pi_1(\Omega)) \cdot \mathrm{m}(\pi_2(\Omega)).$$

1.3 3D Projections

The 3D projections case is relatively the same, except this time it actually utilizes the CS inequality (and thus, why it is in this section). Note that this part of the notes is not pivotal to the project itself in Section 2, and can be skipped. This time, let $\pi_1(x_1, x_2, x_3) = (x_2, x_3)$ and so on and so forth for π_2 and π_3. Hence, for the discrete case, we get

$$\begin{aligned}
\#S_N = \sum_x \chi_{S_N}(x) &\le \sum_{x_1, x_2, x_3} \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3)\chi_{\pi_3(S_N)}(x_1, x_2) \\
&= \sum_{x_1, x_2} \chi_{\pi_3(S_N)}(x_1, x_2) \left(\sum_{x_3} \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3) \right).
\end{aligned}$$

Applying the CS inequality,

$$\le \left(\sum_{x_1, x_2} \chi^2_{\pi_3(S_N)}(x_1, x_2) \right)^{\frac{1}{2}} \cdot \left(\sum_{x_1, x_2} \left(\sum_{x_3} \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3) \right)^2 \right)^{\frac{1}{2}}.$$

Between this line and the next, we use the fact that $\chi^2(\vec{x}) = \chi(\vec{x})$ as $\chi(\vec{x})$ either equals 0 or 1 and $0^2 = 0$ and $1^2 = 1$.

$$\begin{aligned}
&= \left(\sum_{x_1, x_2} \chi_{\pi_3(S_N)}(x_1, x_2) \right)^{\frac{1}{2}} \cdot \left(\sum_{x_1, x_2} \left(\sum_{x_3} \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3) \right)^2 \right)^{\frac{1}{2}} \\
&= \#\pi_3(S_N) \cdot \left(\sum_{x_1, x_2} \left(\sum_{x_3} \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3) \right)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Now don't be intimidated by the large amount of variables and letters in this next line. All we are doing is adding in a new variable when we square the inside sum. Then, the goal from here is to split up the sum by *separating* these many variables.

$$\begin{aligned}
&= \#\pi_3(S_N) \cdot \left(\sum_{x_1,x_2} \sum_{x_3,x_3'} \chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3)\chi_{\pi_1(S_N)}(x_2,x_3')\chi_{\pi_2(S_N)}(x_1,x_3') \right)^{\frac{1}{2}} \\
&\leq \#\pi_3(S_N) \cdot \left(\sum_{x_1,x_2} \sum_{x_3,x_3'} \chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3') \right)^{\frac{1}{2}} \\
&= \#\pi_3(S_N) \cdot \left(\sum_{x_2,x_3} \chi_{\pi_1(S_N)}(x_2,x_3) \cdot \sum_{x_1,x_3'} \chi_{\pi_2(S_N)}(x_1,x_3') \right)^{\frac{1}{2}} \\
&= \sqrt{\#\pi_1(S_N)} \cdot \sqrt{\#\pi_2(S_N)} \cdot \sqrt{\#\pi_3(S_N)}.
\end{aligned}$$

Again, from here, we could get that $N^{\frac{2}{3}} \leq \max_{i=1,2,3} \#\pi_i(S_N)$. Now we will show the continuous case, which works out exactly the same but with integrals!

Let $\Omega \subset \mathbb{R}^3$, and let $A_i = \sqrt{\text{area}(\pi_i(\Omega))}$. Then,

$$\begin{aligned}
\text{vol}(\Omega) &= \int_{\mathbb{R}^3} \chi_\Omega(x)\,\mathrm{d}x \\
&\leq \iiint \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\chi_{\pi_3(\Omega)}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&= \iint \chi_{\pi_3(\Omega)}(x_1,x_2) \left(\int \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\,\mathrm{d}x_3 \right) \mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}$$

Note that we can swap these integrals like this by Fubini's Theorem! By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\leq \left(\iint \chi^2_{\pi_3(\Omega)}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \right)^{1/2} \left(\iint \left(\int \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\,\mathrm{d}x_3 \right)^2 \mathrm{d}x_1\,\mathrm{d}x_2 \right)^{1/2} \\
&\leq A_3^{1/2} \left(\iiiint \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3)\chi_{\pi_1(\Omega)}(x_2,x_3')\chi_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_3\,\mathrm{d}x_3'\,\mathrm{d}x_1\,\mathrm{d}x_2 \right)^{1/2} \\
&\leq A_3^{1/2} \left(\iiiint \chi_{\pi_1(\Omega)}(x_2,x_3)\chi_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_3\,\mathrm{d}x_3'\,\mathrm{d}x_1\,\mathrm{d}x_2 \right)^{1/2} \\
&= A_3^{1/2} \left(\iint \chi^2_{\pi_1(\Omega)}(x_2,x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3 \right)^{1/2} \left(\iint \chi^2_{\pi_2(\Omega)}(x_1,x_3')\,\mathrm{d}x_1\,\mathrm{d}x_3' \right)^{1/2} = A_1^{1/2} A_2^{1/2} A_3^{1/2}.
\end{aligned}$$

Hence, we achieve the nice inequality:

$$\text{vol}(\Omega) \leq \sqrt{\text{area}(\pi_1(\Omega))} \cdot \sqrt{\text{area}(\pi_2(\Omega))} \cdot \sqrt{\text{area}(\pi_3(\Omega))}.$$

□

Question 2. *Is there a nice geometric argument similar to the 2D case that we can use to prove this inequality?*

2 Sobolev's Inequality

We start with a basic concept: if the derivative of a function is 0, then that function is a constant. But what does it mean if the derivative of a function is approximately 0? Is that function approximately a constant? And how would we describe the derivative of a function being *nearly* 0?

Well, consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in C^1$, and $\int_{-\infty}^{\infty} |f'(x)| \, \mathrm{d}x \leq 1$. What can we say about f? Well, using the Fundamental Theorem of Calculus, given $-\infty < a, b < \infty$,

$$\begin{aligned} |f(b) - f(a)| &= \left| \int_a^b f'(x) \, \mathrm{d}x \right| \\ &\leq \int_a^b |f'(x)| \, \mathrm{d}x \\ &\leq \int_{-\infty}^{\infty} |f'(x)| \, \mathrm{d}x \leq 1. \end{aligned}$$

Hence, for all $-\infty < a, b < \infty$, $|f(b) - f(a)| \leq 1$.

How does this concept transfer over to higher dimensions? Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \in C^1$ and $\int_{\mathbb{R}^2} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} \leq 1$. Is it true that $|f(\vec{b}) - f(\vec{a})| \leq 1$ for all $\vec{a}, \vec{b} \in \mathbb{R}^2$? No. Here is a counter example: consider the function

$$f_\epsilon(\vec{x}) = \begin{cases} 0 & |\vec{x}| > 2\epsilon \\ 1 & \vec{x} = 0 \end{cases}$$

and let f_ϵ smoothly interpolate between 0 and 1 for all values of $\vec{x}$ such that $0 < |\vec{x}| < 2\epsilon$. Then, ∇f is supported in the region where $0 < |\vec{x}| < 2\epsilon$, and $|\nabla f| \leq \frac{2}{\epsilon}$. Hence,

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} &= \int_{0 \leq |x| \leq 2\epsilon} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} + \int_{|x| > 2\epsilon} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} \\ &= \int_{0 \leq |x| \leq 2\epsilon} |\nabla f(\vec{x})| \, \mathrm{d}\vec{x} \\ &= \text{area}(Circle) \cdot \frac{2}{\epsilon} \\ &\leq C\epsilon \xrightarrow{\epsilon \to 0} 0 \end{aligned}$$

and where $C > 0$. However, it is not necessarily the case that $|f(\vec{b}) - f(\vec{a})| \leq 1$ for all $\vec{a}, \vec{b} \in \mathbb{R}^2$. In other words, though the "derivative" tends to 0, the difference $|f(\vec{b}) - f(\vec{a})|$ does not.

Hence, from here we want to try and prove something weaker. Suppose that $f \in C^1$ and f is compactly supported.

Remark 3. *Recall that a function is compactly supported if there exists some finitely sized rectangle such that for every (x, y) outside of this rectangle in $\mathbb{R}^2$, $f(x, y) = 0$. For our purposes, I will be picturing a square centered at the origin instead of just "some rectangle", though this is logically equivalent.*

Note 4

For now, I will denote $L^p(\mathbb{R}^2)$ as simply p, i.e.,

$$||f||_{L^p(\mathbb{R}^2)} := ||f||_p = \left(\int_{\mathbb{R}^2} |f(\vec{x})|^p \, \mathrm{d}\vec{x} \right)^{\frac{1}{p}}.$$

I will use the same notation when dealing with $L^p(\mathbb{R}^n)$, though this should be clear from the context.

Now the question is: can we find an upper bound for $||f||_p$ with respect to $||\nabla f||_1$?

2.1 Rescaling Arguments

We can instead ask a broader question than this which will prove helpful: can we find an upperbound for $||f||_p$ with respect to $||\nabla f||_1$? To start answering this question, we can ask: for which p and α does there exist a $C > 0$ such that

$$F(f, p, \alpha) := \frac{||f||_p}{||\nabla f||_1^\alpha} \leq C$$

for all compactly supported $f \in C^1$?

For this, we can use *rescaling arguments* to prove that $p = 2$ and $\alpha = 1$. Assume, for the sake of contradiction that there exists such a $C > 0$ for all f, α, and p (with $f \in C^1$). Then, let $0 < \beta \in \mathbb{R}$ and consider the following:

It should be the case that

$$F(\beta f, p, \alpha) = \frac{||\beta f||_p}{||\nabla \beta f||_1^\alpha} = \frac{\beta}{\beta^\alpha} \frac{||f||_p}{||f||_1^\alpha} \leq \frac{\beta}{\beta^\alpha} \cdot C.$$

Hence, $\alpha = 1$, as if $\alpha > 1$, letting $\beta \to 0$ implies there doesn't exist a finite bound for $F(\beta f, p, \alpha)$, and if $\alpha < 1$ then letting $\beta \to 0$ implies $C = 0$ (clearly not the case by considering any nontrivial example).

Hence, we can now let $\alpha = 1$, and we can do a similar rescaling argument to show that $p = 2$.
Let $g(x, y) = f(\beta x, \beta y)$. Then,

$$\begin{aligned} ||g||_p &= \left(\int_{\mathbb{R}} \int_{\mathbb{R}} (g(x,y))^p \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}} \int_{\mathbb{R}} (f(\beta x, \beta y))^p \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}}. \end{aligned}$$

Letting $u = \beta x$ and $v = \beta y$,

$$\begin{aligned} &= \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\beta^2} \cdot (f(u,v))^p \, \mathrm{d}u \, \mathrm{d}v \right)^{\frac{1}{p}} \\ &= \frac{1}{\beta^{\frac{2}{p}}} ||f||_p. \end{aligned}$$

Similarly,

$$\begin{aligned} ||\nabla g||_1 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \sqrt{(\partial_x g(x,y))^2 + (\partial_y g(x,y))^2} \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \sqrt{(\partial_x f(\beta x, \beta y))^2 + (\partial_y f(\beta x, \beta y))^2} \, \mathrm{d}x \, \mathrm{d}y. \end{aligned}$$

Letting $u = \beta x$ and $v = \beta y$,

$$\begin{aligned} &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\beta^2} \cdot \sqrt{(\beta \partial_u f(u,v))^2 + (\beta \partial_v f(u,v))^2} \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\beta} \cdot \sqrt{(\partial_u f(u,v))^2 + (\partial_v f(u,v))^2} \, \mathrm{d}u \, \mathrm{d}v \\ &= \frac{1}{\beta} \cdot ||\nabla f||_1. \end{aligned}$$

Therefore,

$$\frac{||g||_p}{||\nabla g||_1} = \beta^{1 - \frac{2}{p}} \frac{||f||_p}{||\nabla f||_1} \leq \beta^{1 - \frac{2}{p}} \cdot C.$$

By a similar argument for α, this directly implies p can only be 2. Hence, if there exists a C such that $F(f, p, \alpha) \leq C$, then $p = 2$ and $\alpha = 1$. Now the question is: given $p = 2$ and $\alpha = 1$, does there exist such an upper bound?

This rescaling argument allows us to reframe our main goal of this project.

Remark 5. *Note that as we assume $f \in C^1$ and compactly supported, f must be bounded.*

> **Problem 6**
> Let $f(x, y) \in C^1$ be a compactly supported function, such that $|f(x, y)| \leq 1$ for all $(x, y) \in \mathbb{R}^2$ and $f(x, y) = 0$ for $(x, y) \notin [-1, 1] \times [-1, 1]$. Let there exist an $\vec{x}$ such that $|f(\vec{x})| = 1$. Show there exists an upper bound (C) for $\frac{||f||_2}{||\nabla f||_1}$.

This is equivalent to our previous problem, as previously we assumed $f \in C^1$ was compactly supported, and thus bounded. Therefore, we can rescale f to have upper bound 1, and to be 0 outside of a given square centered at the origin (in this case, $[-1, 1] \times [-1, 1]$).However, this part is just semantics and will simply be a corollary when we solve Problem 6, so we will revisit in the Concluding Remarks.

Question 7. *Can a rescaling argument be used to start to generalize this problem to functions with more variables? Is it always the case that there is only one such scaling invariant L^p space (i.e., will there always only exist one such p and α)? What if we were instead given $||\nabla f||_q$ instead of $||\nabla f||_1$?*

Let's answer these questions! We will answer the most general version of this question.

Higher Dimensions, with $||\nabla \mathbf{f}||_{\mathbf{q}}$: Let $f : \mathbb{R}^n \to \mathbb{R}$, and let $\beta > 0$. Assume that there exists a fixed $C > 0$, p, and α such that for all $f \in C^1$ with f compactly supported such that $\frac{||f||_p}{||\nabla f||_q^\alpha} < C$. We will find values for both p and α.

Consider $g(\vec{x}) = \beta f(\vec{x})$. Then,

$$\frac{||g||_p}{||\nabla g||_q^\alpha} = \frac{||\beta f||_p}{||\nabla \beta f||_q^\alpha} = \beta^{1-\alpha} \cdot \frac{||f||_p}{||\nabla f||_q^\alpha} \leq \beta^{1-\alpha} \cdot C \implies \alpha = 1.$$

Of course, the more interesting part of this problem will be what p is. Let $g(\vec{x}) = f(\beta \vec{x})$. From the last part, we have that $||g||_p = \beta^{-\frac{n}{p}} \cdot ||f||_p$. Now we consider $||\nabla g||_q$.

$$\begin{aligned}
||\nabla g||_q &= \left(\int_{\mathbb{R}^n} \left(\sqrt{\sum_{i=1}^n \partial_{x_i}^2 g(\vec{x})} \right)^q \mathrm{d}\vec{x} \right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}^n} \left(\sqrt{\sum_{i=1}^n \partial_{x_i}^2 f(\beta \vec{x})} \right)^q \mathrm{d}\vec{x} \right)^{\frac{1}{q}}.
\end{aligned}$$

Let $u_i = \beta \, x_i$. Hence, $d\vec{u} = \beta^n \, \mathrm{d}\vec{x}$, and $\partial_{x_i}^2 f(\beta \vec{x}) = \beta^2 \partial_{u_i}^2 f(\vec{u})$. Therefore,

$$\begin{aligned}
&= \left(\frac{1}{\beta^n} \cdot \int_{\mathbb{R}^n} \left(\sqrt{\sum_{i=1}^n \beta^2 \partial_{u_i}^2 f(\vec{u})} \right)^q \mathrm{d}\vec{u} \right)^{\frac{1}{q}} \\
&= \beta^{1-\frac{n}{q}} \cdot \left(\int_{\mathbb{R}^n} \left(\sqrt{\sum_{i=1}^n \partial_{u_i}^2 f(\vec{u})} \right)^q \mathrm{d}\vec{u} \right)^{\frac{1}{q}} \\
&= \beta^{1-\frac{n}{q}} \cdot ||\nabla f||_q.
\end{aligned}$$

Hence,

$$\frac{||g||_p}{||\nabla g||_q} = \beta^{\frac{n}{q} - \frac{n}{p} - 1} \cdot \frac{||f||_1}{||\nabla f||_q} \leq \beta^{\frac{n}{q} - \frac{n}{p} - 1} \cdot C.$$

Therefore, $p = \frac{nq}{n-q}$. This agrees with what we have shown previously with $q = 1$ and $n = 2$.

Question 8. *What can we do if given both $||\nabla f||_{q_1}$ and $||\nabla f||_{q_2}$ for $q_1 \neq q_2$? What happens if $q = n$?*

2.2 Bounding $||f||_2$

Now we actually want to try and bound the ratio

$$\frac{||f||_2}{||\nabla f||_1} = \frac{\left(\int_{\mathbb{R}^2} |f(\vec{x})|^2 \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^2} \sqrt{\partial_x^2(f(x,y)) + \partial_y^2(f(x,y))}\,\mathrm{d}x\,\mathrm{d}y}.$$

To do so, we can try to find an upper bound for $||f||_2$ and find a lower bound to $||\nabla f||_1$. In this subsection, we will try to find a bound for $||f||_2$.

First, we can split $\mathbb{R}^2$ into disjoint subsets in which we know the values of f. Consider the sets

$$V_{2^{-k}} := \{\vec{x} \mid 2^{-k} \geq |f(\vec{x})| > 2^{-k-1}\}.$$

Since $|f(\vec{x})| \leq 1$, we have the following:

$$\begin{aligned} ||f||_2 &= \left(\int_{\mathbb{R}^2} |f(\vec{x})|^2 \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}} \\ &= \left(\sum_{k=0}^{\infty} \int_{V_{2^{-k}}} |f(\vec{x})|^2 \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}}. \end{aligned}$$

Hence, taking the maximum over each set, we get

$$\leq \left(\sum_{k=0}^{\infty} 2^{-2k} \cdot \mathrm{area}(V_{2^{-k}})\right)^{\frac{1}{2}}.$$

Can we find an upper bound to the area part of this inequality? Would that result in some nice inequality? Note that in the end finding an upper bound to the area won't actually pan out. If the reader prefers, they can skip ahead to the next subsection.

We can consider similarly defined sets such that V_λ is a subset of them, and thus has less area than them. Consider the sets

$$U_\lambda := \begin{cases} \{\vec{x} \mid f(\vec{x}) \geq \lambda\} & \lambda > 0 \\ \{\vec{x} \mid f(\vec{x}) \leq \lambda\} & \lambda < 0 \end{cases}.$$

Note the following:

$$\begin{aligned} ||\nabla f||_1 &= \int_{\mathbb{R}^2} |\nabla f(\vec{x})| \,\mathrm{d}\vec{x} \\ &= \int_{\mathbb{R}^2} \sqrt{(\partial_x f(\vec{x}))^2 + (\partial_y f(\vec{x}))^2} \,\mathrm{d}\vec{x} \\ &\geq \int_{\mathbb{R}^2} \sqrt{(\partial_x f(\vec{x}))^2} \,\mathrm{d}\vec{x} \\ ||\nabla f||_1 &\geq \int_{\mathbb{R}^2} |\partial_x f(\vec{x})| \,\mathrm{d}\vec{x}. \end{aligned}$$

Similarly,

$$||\nabla f||_1 \geq \int_{\mathbb{R}^2} |\partial_y f(\vec{x})| \,\mathrm{d}\vec{x}.$$

Let's consider the first of these two inequalities. Using the method we used for single variable functions (with the Fundamental Theorem of Calculus), we can obtain the following: For all $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned} |f(x_1, y) - f(x_2, y)| &= \left| \int_{x_1}^{x_2} \partial_x f(x, y) \, \mathrm{d}x \right| \\ &\leq \int_{x_1}^{x_2} |\partial_x f(x, y)| \, \mathrm{d}x \\ &\leq \int_{\mathbb{R}} |\partial_x f(x, y)| \, \mathrm{d}x. \end{aligned}$$

Integrating both sides of this inequality, we get that for all $x_1, x_2 \in \mathbb{R}$ (which may depend on y)

$$\int_{\mathbb{R}} |f(x_1, y) - f(x_2, y)| \, \mathrm{d}y \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_x f(x, y)| \, \mathrm{d}x \, \mathrm{d}y \leq ||\nabla f||_1. \tag{3}$$

Replacing x with y, we can similarly get that for all $y_1, y_2 \in \mathbb{R}$ (which may depend on x),

$$\int_{\mathbb{R}} |f(x, y_1) - f(x, y_2)| \, \mathrm{d}x \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_y f(x, y)| \, \mathrm{d}y \, \mathrm{d}x \leq ||\nabla f||_1. \tag{4}$$

Furthermore, as $U_\lambda \subset \mathbb{R}^2$, we have that

$$\operatorname{area}(U_\lambda) \leq \mathrm{m}(\pi_1(U_\lambda)) \cdot \mathrm{m}(\pi_2(U_\lambda)). \tag{5}$$

Let $y \in \pi_1(U_\lambda)$. Now we are going to use equation (4). Since $y \in \pi_1(U_\lambda)$, there exists an $x_2 \in \mathbb{R}$ such that $x_2 = \sup\{x \in \pi_2(U_\lambda) \mid f(x, y) = \lambda\}$. Furthermore, since f is compactly supported, there exists an $x_1 < x_2 \in \mathbb{R}$ such that $f(x_1, y) = 0$. Hence, using these values of x_1 and x_2 for each $y \in \pi_1(U_\lambda)$ in equation (2.1), we get that

$$\begin{aligned} ||\nabla f||_1 &\geq \int_{\pi_1(U_\lambda)} |f(x_1, y) - f(x_2, y)| \, \mathrm{d}y \\ &\geq \int_{\pi_1(U_\lambda)} |\lambda| \, \mathrm{d}y \\ &= |\lambda| \cdot \mathrm{m}(\pi_1(U_\lambda)) \implies \mathrm{m}(\pi_1(U_\lambda)) \leq \frac{||\nabla f||_1}{|\lambda|}. \end{aligned}$$

Similarly,

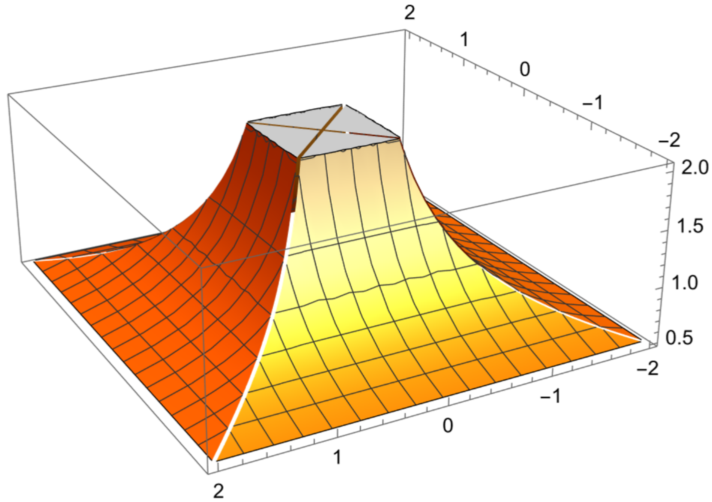
$$1 \geq |\lambda| \cdot \mathrm{m}(\pi_2(U_\lambda)) \implies \mathrm{m}(\pi_2(U_\lambda)) \leq \frac{||\nabla f||_1}{|\lambda|}.$$

Hence, applying equation (6), we have that

$$\operatorname{area}(U_\lambda) \leq \frac{||\nabla f||_1^2}{\lambda^2}. \tag{6}$$

Question 9. *Is this inequality approximately sharp?*

For simplicity, let $||\nabla f||_1 = 1$ to visualize this example, though note this would easily translate over $||\nabla f|| = c$ for any nonnegative constant c. Consider the following example: $f(x, y) = \frac{1}{\max\{|x|, |y|\}}$.



Thus, U_λ is a square of side-length $\frac{1}{\lambda}$, and thus $\text{area}(U_\lambda) = \frac{1}{\lambda^2}$ for all $\lambda \neq 0$. However, f is neither continuous (discontinuous at the origin) nor compactly supported. How we can "fix" this example? In the region where $R \leq \max\{|x|, |y|\} \leq 2R$, let f smoothly interpolate to 0, and let f be 0 if $\max\{|x|, |y|\} \geq 2R$. This lets f be compactly supported, and we can make R arbitrarily large (though finite), implying that $\text{area}(U_\lambda) \leq \frac{1}{\lambda^2}$ will be approximately sharp, or exactly sharp, almost everywhere when R is sufficiently large. Addressing the discontinuity at the origin, in the region of the region where $0 \leq \max\{|x|, |y|\} \leq \frac{1}{R}$ (related to the R in fixing compactly supported), let $f(\vec{x})$ smoothly interpolate to the value R.

After making these changes to f, it seems to be the case that this works as a sufficient counterexample to show that $\text{area}(U_\lambda) \leq \frac{||\nabla f||_1^2}{\lambda^2}$ is approximately sharp.

So, given this inequality is approximately sharp, can we derive some sort of connection between $\text{area}(U_\lambda) \leq \frac{||\nabla f||_1^2}{\lambda^2}$ and $||f||_2$?

Conjecture 10

Perhaps we can say something along the lines of

$$\left(\int_{\mathbb{R}^2} |f(\vec{x})|^2 \,\mathrm{d}\vec{x}\right)^{\frac{1}{2}} \leq ||\nabla f||_1^2 \iff \forall \lambda \neq 0,\ \text{area}(U_\lambda) \leq \frac{||\nabla f||_1^2}{\lambda^2}.$$

Well we can certainly prove the forward direction.

$$\begin{aligned}
||\nabla f||_1^2 &\geq \int_{\mathbb{R}^2} |f(\vec{x})|^2 \,\mathrm{d}\vec{x} \\
&\geq \int_{U_\lambda} |f(\vec{x})|^2 \,\mathrm{d}\vec{x} \\
&\geq \int_{U_\lambda} |\lambda|^2 \,\mathrm{d}\vec{x}.
\end{aligned}$$

Hence,

$$\text{area}(U_\lambda) = \int_{U_\lambda} 1 \,\mathrm{d}\vec{x} \leq \frac{||\nabla f||_1^2}{\lambda^2}.$$

This is known as Chebyshev's inequality, which comes up in probability.

But in trying to prove the other direction, how can we use $\text{area}(U_\lambda) \leq \frac{||\nabla f||_1^2}{\lambda^2}$ to approximate this integral? Well, this goes back to the initial part of this section: finding an upper bound for $||f||_2$. We could certainly try and plug in the area inequality. However, this will give us a divergent series, as shown below:

$$
\begin{aligned}
||f||_2 &\le \left(\sum_{k \in \mathbb{N}} 2^{-2(k-1)} \cdot \text{area}(V_{2^{-(k-1)}}) \right)^{\frac{1}{2}} \\
&\le \left(\sum_{k \in \mathbb{N}} 2^{-2(k-1)} \cdot \text{area}(U_{2^{-(k-1)}}) \right)^{\frac{1}{2}} \\
&\le \left(\sum_{k \in \mathbb{N}} 2^{-2(k-1)} \cdot \frac{||\nabla f||_1^2}{2^{-2(k-1)}} \right)^{\frac{1}{2}} \\
&= \left(\sum_{k \in \mathbb{N}} ||\nabla f||_1^2 \right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence, unless $||\nabla f||_1 = 0$ (which implies f is constant and thus $||f||_2 = 0$ as f is compactly supported), we get a divergent series. Thus, this approach of finding an upper bound for the area doesn't quite pan out. Therefore, we hope to (and will) instead find a lower bound for $||\nabla f||_1$ that depends on the areas of V_λ and hope that things cancel out in the end (they will).

2.3 Bounding $||\nabla f||_1$

Instead, we can try to find a lower bound for $||\nabla f||_1$ without assuming $||\nabla f||_1 \le 1$. In trying to find an upper bound to the ratio $\frac{||f||_2}{||\nabla f||_1}$, we don't assume anything about the value of $||\nabla f||_1$, but the inequality $||\nabla f||_1 \le 1$ was crucial in first proving $\text{area}(U_\lambda) \le \frac{1}{\lambda^2}$, which we ultimately can't use [as again, it leads to a divergent series]. Hence, we will *not* be using this inequality, though it was useful to start exploring this problem. We have the following.

$$
\begin{aligned}
||\nabla f||_1 &= \int_{\mathbb{R}^2} \sqrt{\partial_x^2 f(\vec{x}) + \partial_y^2 f(\vec{x})}\, \mathrm{d}\vec{x} \\
&\ge \int_{\mathbb{R}^2} |\partial_x f(\vec{x})|\, \mathrm{d}\vec{x} \\
&\ge \sum_{k \text{ even}} \int_{V_{2^{-k}} \cup V_{2^{-k-1}}} |\partial_x f(x,y)|\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}
$$

Recall that we assume there exists a $\vec{x}$ such that $|f(\vec{x})| = 1$, and thus we can state the last line. Then,

$$
\begin{aligned}
||\nabla f||_1 &\ge \sum_{k \text{ even}} \int_{V_{2^{-k}} \cup V_{2^{-k-1}}} |\partial_x f(x,y)|\, \mathrm{d}x\, \mathrm{d}y \\
&= \sum_{k \text{ even}} \int_{\pi_1(V_{2^{-k}} \cup V_{2^{-k-1}})} \int_{\{x | (x,y) \in (V_{2^{-k}} \cup V_{2^{-k-1}})\}} |\partial_x f(x,y)|\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}
$$

Similarly, we can get that

$$
||\nabla f||_1 \ge \sum_{k \text{ odd}} \int_{\pi_1(V_{2^{-k}} \cup V_{2^{-k-1}})} \int_{\{x | (x,y) \in (V_{2^{-k}} \cup V_{2^{-k-1}})\}} |\partial_x f(x,y)|\, \mathrm{d}x\, \mathrm{d}y.
$$

Adding these two inequalities together, we get that

$$
\begin{aligned}
||\nabla f||_1 &\ge \frac{1}{2} \sum_{k=0}^{\infty} \int_{\pi_1(V_{2^{-k}} \cup V_{2^{-k-1}})} \int_{\{x | (x,y) \in (V_{2^{-k}} \cup V_{2^{-k-1}})\}} |\partial_x f(x,y)|\, \mathrm{d}x\, \mathrm{d}y \\
&\ge \frac{1}{2} \sum_{k=0}^{\infty} \int_{\pi_1(V_{2^{-k}})} \int_{\{x | (x,y) \in (V_{2^{-k}} \cup V_{2^{-k-1}})\}} |\partial_x f(x,y)|\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}
$$

We can pick $x_{1,k}$ and $x_{2,k}$ such that $f(x_{1,k}, y) = 2^{-k-2}$ and $f(x_{2,k}, y) = 2^{-k-1}$. I will explain at the end of this proof why we can do this. By the Fundamental Theorem of Calculus,

$$\begin{aligned}
||\nabla f||_1 &\geq \frac{1}{2}\sum_{k=0}^{\infty}\int_{\pi_1(V_{2^{-k}})}\int_{\{x|(x,y)\in(V_{2^{-k}}\cup V_{2^{-k-1}})\}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \sum_{k=0}^{\infty}\int_{\pi_1(V_{2^{-k}})}\int_{x_{1,k}}^{x_{2,k}} |\partial_x f(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\int_{\pi_1(V_{2^{-k}})} \frac{1}{4}\cdot 2^{-k}\,\mathrm{d}x\,\mathrm{d}y \\
||\nabla f||_1 &\geq \frac{1}{8}\sum_{k=0}^{\infty} 2^{-k}\cdot \mathrm{m}(\pi_1(V_{2^{-k}})).
\end{aligned}$$

We can similarly do this with π_2 instead of π_1. Then we get the inequality

$$||\nabla f||_1 \geq \frac{1}{8}\sum_{k=0}^{\infty} 2^{-k}\cdot \mathrm{m}(\pi_2(V_{2^{-k}})).$$

Therefore,

$$||\nabla f||_1^2 \geq \frac{1}{16}\sum_{k=0}^{\infty} 2^{-2k}\cdot \mathrm{m}(\pi_1(V_{2^{-k}}))\cdot \mathrm{m}(\pi_2(V_{2^{-k}})) \geq \frac{1}{16}\sum_{k=0}^{\infty} 2^{-2k}\cdot \mathrm{area}(V_{2^{-k}})$$

using equation (2).

There is just that one caveat that we need to prove on. In order to use the Fundamental Theorem of Calculus to find a lower bound for $||\nabla f||_1$, we need to show that $x_{1,k}$ and $x_{2,k}$ are connected by a line segment in $V_{2^{-k}} \cup V_{2^{-k-1}}$ when $y \in \pi_1(V_{2^{-k}})$.

This is why we went through the trouble of splitting the sum into even and odd parts– so that this step of the proof worked out the way we wanted to. Let $y \in \pi_1(V_{2^{-k}})$. Since this function is compactly supported, we know that there exists an x such that $f(x, y) = 2^{-k-1}$ and thus $(x, y) \in V_{2^{-k}} \cup V_{2^{-k-1}}$. Let $x_{2,k}$ be the smallest such x that satisfies $f(x_{2,k}, y) = 2^{-k-1}$.

Since f is compactly supported, this implies that for all $x \leq x_{2,k}$, $f(x, y) \leq 2^{-k-1}$, as otherwise we would have a smaller $x_{2,k}$. Hence, there must exist an $x_{1,k}$ in the set

$$\{x \in \mathbb{R} \mid x \leq x_{2,k} \text{ and } (x, y) \in V_{2^{-k-1}}\}$$

with $f(x_{1,k}, y) = 2^{-k-2}$ such that $[x_{1,k}, x_{2,k}] \times y$ is a connected line segment in $V_{2^{-k}} \cup V_{2^{-k-1}}$. Therefore, we were in fact able to apply the Fundamental Theorem of Calculus.

Hence,

$$||\nabla f||_1 \geq \frac{1}{4}\left(\sum_{k=0}^{\infty} 2^{-2k}\cdot \mathrm{area}(V_{2^{-k}})\right)^{\frac{1}{2}}.$$

This looks familiar right? This looks almost the same as the upper bound we got for $||f||_2$. Therefore,

$$\frac{||f||_2}{||\nabla f||_1} \leq \frac{\left(\sum_{k=0}^{\infty} 2^{-2k}\cdot \mathrm{area}(V_{2^{-k}})\right)^{\frac{1}{2}}}{\frac{1}{4}\left(\sum_{k=0}^{\infty} 2^{-2k}\cdot \mathrm{area}(V_{2^{-k}})\right)^{\frac{1}{2}}} = 4.$$

3 Concluding Remarks

So at this point, we have proven the following theorem:

Theorem 11
Let $f(x, y) \in C^1$ be a compactly supported function, such that $|f(x, y)| \leq 1$ for all $(x, y) \in \mathbb{R}^2$ and $f(x, y) = 0$ for all $(x, y) \notin [-1, 1] \times [-1, 1]$. Then,

$$\frac{||f||_2}{||\nabla f||_1} \leq 4.$$

We can very easily generalize this though for all two-dimensional continuous compactly-supported functions in general:

Corollary 12
Let $f(x, y) \in C^1$ be a compactly supported function, such that $|f(x, y)| \leq Z$ for all $(x, y) \in \mathbb{R}^2$ and $f(x, y) = 0$ for all $(x, y) \notin [-R, R] \times [-R, R]$. Then, we have

$$\frac{||f||_2}{||\nabla f||_1} \leq 4.$$

Proof: Assume Z is nonzero, as then this is trivially true. Consider the function $g(x, y) = \frac{1}{Z} \cdot f(Rx, Ry)$. Hence, $g(x, y) \in C^1$, $|g(x, y)| \leq 1$ for all $(x, y) \in \mathbb{R}^2$, and $g(x, y) = 0$ for all $(x, y) \notin [-1, 1] \times [-1 \times 1]$. Therefore, based on Theorem 10,

$$\frac{||g||_2}{||\nabla g||_1} \leq 4.$$

We can use this to find $\frac{||f||_2}{||\nabla f||_1}$:

$$\begin{aligned}
4 &\geq \frac{||g||_2}{||\nabla g||_1} \\
&= \frac{\left(\int_{\mathbb{R}^2} |g(\vec{x})|^2 \, \mathrm{d}\vec{x}\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^2} |\nabla g(\vec{x})| \, \mathrm{d}\vec{x}} \\
&= \frac{\left(\int_{\mathbb{R}^2} |\frac{1}{Z} \cdot f(R\vec{x})|^2 \, \mathrm{d}\vec{x}\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^2} |\nabla \frac{1}{Z} \cdot f(R\vec{x})| \, \mathrm{d}\vec{x}} \\
&= \frac{\left(\int_{\mathbb{R}^2} |f(R\vec{x})|^2 \, \mathrm{d}\vec{x}\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^2} |\nabla f(R\vec{x})| \, \mathrm{d}\vec{x}}.
\end{aligned}$$

This actually shows something pretty interesting; the upper bound on $|f|$ actually plays no effect on the ratio $\frac{||f||_2}{||\nabla f||_1}$, though perhaps this is inherently clear from the rescaling arguments earlier in this chapter. In any case, let $\vec{u} = R\vec{x}$ such that $\mathrm{d}\vec{u} = R^2 \, \mathrm{d}\vec{x}$. Hence,

$$\begin{aligned}
&= \frac{\left(\int_{\mathbb{R}^2} \frac{1}{R^2} \cdot |f(\vec{u})|^2 \, \mathrm{d}\vec{u}\right)^{\frac{1}{2}}}{\int_{\mathbb{R}^2} \frac{R}{R^2} \cdot |\nabla f(\vec{u})| \, \mathrm{d}\vec{u}} \\
&= \frac{||f||_2}{||\nabla f||_1}.
\end{aligned}$$

$\square$